Correction to the Fukuda–Kawata Young's modulus theory and the Fukuda–Chou strength theory for short fibre-reinforced composite materials

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The theories for modulus and strength of short fibre-reinforced composite materials are based on the calculation of the force sustained by fibres crossing an arbitrary line perpendicular to the applied load, called the scan line, in a thin, rectangular specimen. The widely referenced Fukuda–Kawata modulus theory and the Fukuda–Chou strength theory are based on an apparently incorrect procedure for the calculation of the force sustained by the fibres crossing the scan line. The error is explained in detail by comparing the Fukuda–Kawata modulus theory and the Cox modulus theory. The magnitude of this error is calculated for specific cases.

1. Introduction

The pioneering research of Cox [1] on the elasticity and strength of paper and other fibrous materials serves as the foundation of two slightly divergent fields: paper physics and mechanics of fibre-reinforced composite materials. Cox [1] intended his original contribution to be applicable to materials that derive their stiffness and strength from thin fibres. Researchers in the fields of paper [2–6] and composite materials [7, 8] have used the ideas of Cox [1] as basic building blocks to advance improved theories for the Young's modulus and strength of these fibrous materials.

Kallmes and Perez [2] developed a network model for the Young's modulus and the tensile strength of paper in 1966. Kallmes *et al.* [3–6] expanded the original model in 1978 and it is considered to be one of the major models for strength in the paper physics area today. This model was based on the concept of an ideal sheet of paper with variable fibre orientation as postulated by Cox [1], together with the behaviour of inter-fibre bonds and the definitions of the critical events in the sheet: fibre failure and bond failure. The fibre length distribution was not considered in this model.

Fukuda and Kawata [7] developed a theory for the Young's modulus of short-fibre reinforced composites with variable fibre length and orientation in 1974. Fukuda and Chou [8] adopted the basic probabilistic approach of Fukuda and Kawata [7] to develop a theory for the strength of short-fibre reinforced composites with variable fibre length and orientation in 1982. These theories are extensively cited by the composites research community as these were the first to consider fibre length distribution and fibre orientation distribution together to derive expressions for the Young's modulus and strength of short fibre-reinforced composite materials. The theories were formulated in terms of fibre length and fibre orientation distribution functions and the physical properties of the fibre, the matrix and the composite material. The final expressions were presented as a modified "rule of mixtures".

The theory of Cox [1] as well as the improved theories are based on one fundamental point: calculation of the force sustained by the fibres crossing a scan line, i.e. an arbitrary line perpendicular to the applied load in a thin, rectangular specimen. Cox [1], Kallmes and Perez [2] and Kallmes *et al.* [3–6] use a simple and straightforward method, referred to as the paper physics approach here, to calculate the force sustained by the fibres across the scan line. The paper physics approach involves:

(i) calculating the number of fibres in the specimen of length L and orientation θ that cross the scan line,
(ii) finding the axial force developed in a fibre of length L and orientation θ,
(iii) finding the load-direction component of the axial force in a fibre of length L and orientation θ,
(iv) multiplying the number of fibres of length L and orientation θ that cross the scan line by the load-direction component of the axial force in a fibre of length L and orientation θ, and
(v) integrating the above quantity over fibre length and fibre orientation to find the total force sustained by the fibres crossing the scan line.

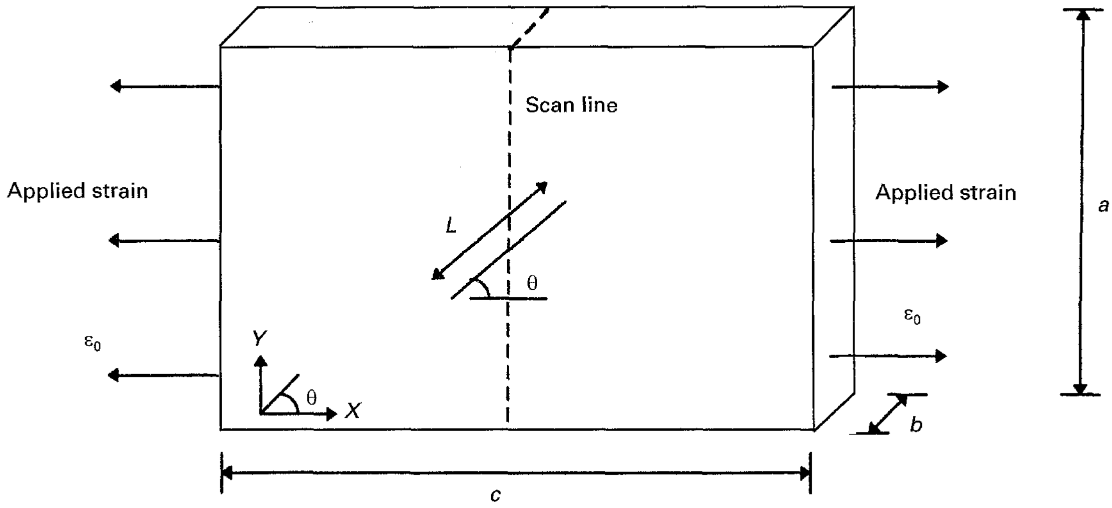

Figure 1 The specimen.

Fukuda and Kawata [7] and Fukuda and Chou [8] utilize an averaging method, referred to as the composite mechanics approach here, to calculate the force sustained by the fibres across the scan line. The composite mechanics approach involves:

(i) calculating the number of fibres in the specimen that cross the scan line,
(ii) finding the axial force developed in a fibre of length L and orientation θ,
(iii) finding the load-direction component of the axial force in a fibre of length L and orientation θ,
(iv) finding the average load-direction component of the axial force in all the fibres in the specimen, and
(v) multiplying the number of fibres that cross the scan line by the average load-direction component of the force in all the fibres in the specimen to find the total force sustained by the fibres crossing the scan line.

There appears to be a mistake in the calculation of the force sustained by the fibres across the scan line in the derivations of Fukuda and Kawata [7] and Fukuda and Chou [8]: the average load-direction component of the axial force is found by averaging over all the fibres in the specimen rather than averaging over the fibres that cross the scan line. Although these articles [7, 8] have been extensively cited, the authors are not aware of any publication dealing with this mistake. This mistake will be explained in detail by comparing the paper physics approach and the composite mechanics approach for the Fukuda–Kawata modulus theory with a case study. The paper physics approach will be shown to be the correct approach. None of the other fundamental assumptions used by Fukuda and Kawata [7] and Fukuda and Chou [8] are contested.

The present article will consider a simple composite specimen subjected to applied strain, similar to the one used by Fukuda and Kawata [7]. The force sustained by the fibres across the scan line in this specimen will be calculated by the paper physics and composite mechanics approaches. The fibre orientation and fibre length factors and hence the force sustained by the fibres crossing the scan line will be shown to be underpredicted by the composite mechanics approach.

The specimen is in the shape of a thin, rectangular plate of dimensions a, b and c with the dimension c parallel to the X-axis direction as shown in Fig. 1. The specimen consists of N straight fibres dispersed in and bonded to a matrix resulting in a fibre volume fraction V_f. The fibres have a circular cross-section with a radius r_f. The fibre orientation is defined as the angle between the axial direction of the fibre and the X-axis direction. A single fibre of length L and orientation θ is shown in Fig. 1. The statistical variation in the length and orientation of the fibres are defined by independent probability density functions $h(L)$ and $g(\theta)$, respectively. Load, in the form of strain ε_c, is applied to the specimen in a direction parallel to the X-axis direction.

2. General theory

The area of cross-section of the fibre is

$$A_f = \pi r_f^2 \tag{1}$$

The two-dimensional probability density function of the fibre orientation distribution satisfies the following condition:

$$\int_0^{\pi/2} g(\theta)\,d\theta = 1 \tag{2}$$

assuming that the distribution of fibre orientation is symmetric with respect to the applied strain along the X-axis direction. The probability density function of the fibre length distribution satisfies the following condition:

$$\int_0^{\infty} h(L)\,dL = 1 \tag{3}$$

The average fibre length is defined as

$$\bar{L} = \int_0^{\infty} Lh(L)\,dL \tag{4}$$

The length of projection on the X-axis of a fibre of length L and orientation θ is

$$L_X = L\cos\theta \tag{5}$$

The volume of the specimen is

$$V = abc \tag{6}$$

and the volume fraction of the fibres can be defined as

$$V_f = \frac{N A_f \bar{L}}{abc} \tag{7}$$

or

$$N = \frac{V_f\, abc}{A_f \bar{L}} \tag{8}$$

The average axial stress in a fibre of length L and orientation θ in the specimen is given by Fukuda and Kawata [7] as

$$\bar{\sigma}_f = \phi E_f \varepsilon_0 (\cos^2\theta - \nu_s \sin^2\theta) \tag{9}$$

where ϕ = a dimensionless function of fibre length and other fibre, matrix and specimen properties, E_f = Young's modulus of the fibre, ε_0 = strain applied to the specimen, and ν_s = Poisson's ratio of the specimen.

The average axial force in a fibre of length L and orientation θ is

$$\begin{aligned}\bar{F}_f &= A_f \bar{\sigma}_f \\ &= A_f \phi E_f \varepsilon_0 (\cos^2\theta - \nu_s \sin^2\theta)\end{aligned} \tag{10}$$

The average axial force in a fibre of length L and orientation θ projected in the direction of the applied strain (X-axis direction) is

$$\begin{aligned}F_x &= \bar{F}_f \cos\theta \\ &= A_f \phi E_f \varepsilon_0 (\cos^2\theta - \nu_s \sin^2\theta)\cos\theta\end{aligned} \tag{11}$$

3. Paper physics approach

The number of fibres of length between L and $(L + \mathrm{d}L)$ and orientation between θ and $(\theta + \mathrm{d}\theta)$ is

$$\begin{aligned}N_{L\theta} &= N h(L)\mathrm{d}L g(\theta)\mathrm{d}\theta \\ &= \frac{V_f\, abc}{A_f \bar{L}} h(L)\mathrm{d}L g(\theta)\mathrm{d}\theta\end{aligned} \tag{12}$$

The total length of projection on the X-axis of the $N_{L\theta}$ fibres is

$$\begin{aligned}L_T &= N_{L\theta} L_X \\ &= \frac{V_f\, abc}{A_f \bar{L}} h(L)\mathrm{d}L g(\theta)\mathrm{d}\theta L\cos\theta\end{aligned} \tag{13}$$

The number of fibres of length between L and $(L + \mathrm{d}L)$ and orientation between θ and $(\theta + \mathrm{d}\theta)$ that cross a scan line is

$$\begin{aligned}N_{\text{scan}} &= \frac{L_T}{c} \\ &= \frac{V_f\, ab}{A_f \bar{L}} h(L)\,\mathrm{d}L g(\theta)\mathrm{d}\theta L\cos\theta\end{aligned} \tag{14}$$

The total load that fibres crossing the scan line carry is

$$\begin{aligned}F_T &= \sum_L \sum_\theta N_{\text{scan}} F_x \\ &= \int_0^\infty \int_0^{\pi/2} \left[\frac{V_f\, ab}{A_f \bar{L}} h(L)\mathrm{d}L g(\theta)\mathrm{d}\theta L\cos\theta\right] \\ &\quad \times [A_f \phi E_f \varepsilon_0 (\cos^2\theta - \nu_s \sin^2\theta)\cos\theta] \\ &= \frac{E_f V_f \varepsilon_0\, ab}{\bar{L}} \int_0^\infty \int_0^{\pi/2} \phi L h(L) g(\theta) \\ &\quad (\cos^4\theta - \nu_s \sin^2\theta\cos^2\theta)\mathrm{d}\theta\,\mathrm{d}L\end{aligned} \tag{15}$$

Since $h(L)$ and $g(\theta)$ are independent of each other, the integrals can be separated as

$$\begin{aligned}F_T &= E_f V_f \varepsilon_0\, ab \left[\int_0^{\pi/2} g(\theta)(\cos^4\theta - \nu_s \sin^2\theta\cos^2\theta)\mathrm{d}\theta\right] \\ &\quad \times \left[\frac{1}{\bar{L}} \int_0^\infty \phi L h(L)\mathrm{d}L\right] \\ &= E_f V_f \varepsilon_0\, ab\, [C_{PP}]\,[D_{PP}]\end{aligned} \tag{16}$$

where C_{PP} is the fibre orientation factor defined as:

$$C_{PP} = \int_0^{\pi/2} g(\theta)(\cos^4\theta - \nu_s \sin^2\theta\cos^2\theta)\mathrm{d}\theta \tag{17}$$

and D_{PP} is the fibre length factor defined as:

$$D_{PP} = \frac{1}{\bar{L}} \int_0^\infty \phi L h(L)\mathrm{d}L \tag{18}$$

4. Composite mechanics approach

The average length of projection on the X-axis of the fibres in the specimen is

$$\bar{L}_X = \int_0^\infty \int_0^{\pi/2} L\cos\theta h(L)\,\mathrm{d}L g(\theta)\,\mathrm{d}\theta \tag{19}$$

Since $h(L)$ and $g(\theta)$ are independent of each other, Equation 19 can be rewritten using Equation 4 as

$$\bar{L}_X = \bar{L} \int_0^{\pi/2} g(\theta)\cos\theta\,\mathrm{d}\theta \tag{20}$$

The average number of fibres that cross the scan line can be found as

$$\bar{N}_{\text{scan}} = \frac{N\bar{L}_X}{c} = \frac{V_f\, ab}{A_f} \int_0^{\pi/2} g(\theta)\cos\theta\,\mathrm{d}\theta \tag{21}$$

$\bar{N}_{\text{scan}}$ is actually the total number of fibres that cross the scan line in the specimen. The average value of the X-direction component of the axial force "for all the fibres in the specimen" can be found as

$$\begin{aligned}\bar{F}_X &= \int_0^\infty \int_0^{\pi/2} [F_X] h(L)\,\mathrm{d}L g(\theta)\,\mathrm{d}\theta \\ &= \int_0^\infty \int_0^{\pi/2} [A_f \phi E_f \varepsilon_0 \\ &\quad \times (\cos^2\theta - \nu_s \sin^2\theta)\cos\theta] h(L)\,\mathrm{d}L g(\theta)\,\mathrm{d}\theta \\ &= A_f E_f \varepsilon_0 \int_0^\infty \int_0^{\pi/2} \phi h(L) g(\theta) \\ &\quad \times (\cos^3\theta - \nu_s \sin^2\theta\cos\theta)\mathrm{d}\theta\,\mathrm{d}L\end{aligned} \tag{22}$$

The total load that fibres crossing the scan line carry is given by Fukuda and Kawata [7] as

$$F_{\mathrm{T}} = [\bar{N}_{\mathrm{scan}}][\bar{F}_X] \quad (23)$$

Thus, the total load is calculated as the product of the number of fibres crossing the scan line and the average projected axial force in a fibre. But, the average projected axial force in a fibre has been computed "for all the fibres in the specimen" rather than for those fibres which actually cross the scan line. As a result of this, the fibres which carry lower loads (shorter fibres or fibres aligned closer to the transverse direction) are weighted too heavily in the computation of the average projected axial force in a fibre, since they are in fact less likely to intersect the scan line. This results in an underprediction of modulus, since the total stress at a given applied strain is underpredicted. The magnitude of the error depends on the fibre orientation distribution, fibre length distribution and the magnitude of the shear lag effect.

Equation 27 can be written as

$$F_{\mathrm{T}} = \left[\frac{V_{\mathrm{f}}\, ab}{A_{\mathrm{f}}}\int_0^{\pi/2} g(\theta)\cos\theta\, \mathrm{d}\theta\right]\left[A_{\mathrm{f}}E_{\mathrm{f}}\varepsilon_0 \int_0^{\infty}\int_0^{\pi/2} \times \phi h(L)\, g(\theta)(\cos^3\theta - \nu_{\mathrm{s}}\sin^2\theta\cos\theta)\,\mathrm{d}\theta\, \mathrm{d}L\right]$$

$$= E_{\mathrm{f}}V_{\mathrm{f}}\varepsilon_0\, ab \int_0^{\pi/2} g(\theta)\cos\theta\,\mathrm{d}\theta \int_0^{\infty}\int_0^{\pi/2} \times \phi h(L) g(\theta)(\cos^3\theta - \nu_{\mathrm{s}}\sin^2\theta\cos\theta)\mathrm{d}\theta\,\mathrm{d}L \quad (24)$$

Since $h(L)$ and $g(\theta)$ are independent of each other, the integrals can be separated as

$$F_{\mathrm{T}} = E_{\mathrm{f}}V_{\mathrm{f}}\varepsilon_0\, ab\left[\int_0^{\pi/2} g(\theta)\cos\theta\,\mathrm{d}\theta \int_0^{\pi/2} g(\theta) \times(\cos^3\theta - \nu_{\mathrm{s}}\sin^2\theta\cos\theta)\mathrm{d}\theta\right]\left[\int_0^{\infty}\phi h(L)\,\mathrm{d}L\right]$$

$$= E_{\mathrm{f}}V_{\mathrm{f}}\varepsilon_0 ab\,[C_{\mathrm{CM}}][D_{\mathrm{CM}}] \quad (25)$$

where C_{CM} is the fibre orientation factor defined as:

$$C_{\mathrm{CM}} = \int_0^{\pi/2} g(\theta)\cos\theta\,\mathrm{d}\theta \int_0^{\pi/2} g(\theta) \times(\cos^3\theta - \nu_{\mathrm{s}}\sin^2\theta\cos\theta)\,\mathrm{d}\theta \quad (26)$$

and D_{CM} is the fibre length factor defined as:

$$D_{\mathrm{CM}} = \int_0^{\infty}\phi h(L)\mathrm{d}L \quad (27)$$

Comparison of Equations 17 and 18 with Equations 26 and 27 clearly brings out the difference in the paper physics and the composite mechanics approaches. Some specific cases are considered in the next two sections to illustrate the magnitude of this difference.

5. Fibre orientation factor

The limiting case of a specimen with very long fibres is considered in this section; the assumption leads to

$$D_{\mathrm{PP}} = D_{\mathrm{CM}} = 1 \quad (28)$$

The fibre orientation factors resulting from the two approaches are now calculated for the following probability density function for fibre orientation distribution:

$$g(\theta) = \frac{1}{\alpha} \quad \text{in } 0 \leqslant \theta \leqslant \alpha$$

$$g(\theta) = 0 \quad \text{in } \theta > \alpha \quad (29)$$

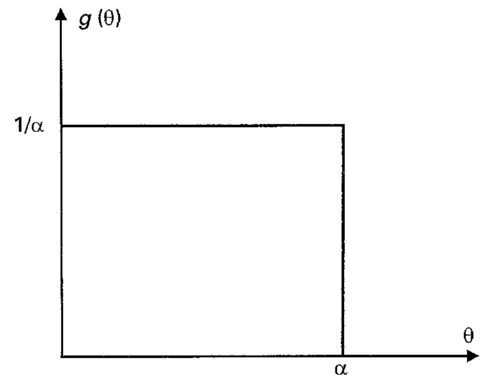

Figure 2 Schematic of the probability density function for fibre orientation distribution.

The function is shown schematically in Fig. 2. The fibre orientation factors C_{PP} and C_{CM} are shown in Fig. 3 as a function of α with the Poisson's ratio of the specimen taken as 1/3. The variation in Poisson's ratio of the specimen with α is ignored for this calculation. As can be seen, the calculation procedure outlined under the composite mechanics approach consistently underpredicts the fibre orientation factor and thus the force sustained by the fibres across the scan line. The value of C_{PP} for a random orientation of fibres is 1/3 which is the same value predicted by Cox [1].

6. Fibre length factor

The limiting case of a specimen with unidirectional orientation of fibres is considered in this section; the assumption leads to

$$C_{\mathrm{PP}} = C_{\mathrm{CM}} = 1 \quad (30)$$

Fukuda and Kawata [7] have predicted ϕ using an advanced analysis for the mechanism of load transfer between a short fibre and matrix. However, it is enough to consider the simple prediction of ϕ given by Cox [1] for the purpose of this discussion concerning a specimen with a unidirectional orientation of fibres. Therefore, from Cox [1],

$$\phi = \left\{1 - \frac{\tanh(\beta L/2)}{(\beta L/2)}\right\} \quad (31)$$

$$\beta = \left[\frac{2\pi G_{\mathrm{m}}}{E_{\mathrm{f}}A_{\mathrm{f}}\ln(R/r_{\mathrm{f}})}\right]^{1/2} = \left\{\frac{2\pi\left[\frac{E_{\mathrm{m}}}{2(1+\nu_{\mathrm{m}})}\right]}{E_{\mathrm{f}}(\pi r_{\mathrm{f}}^2)\ln(R/r_{\mathrm{f}})}\right\}^{1/2}$$

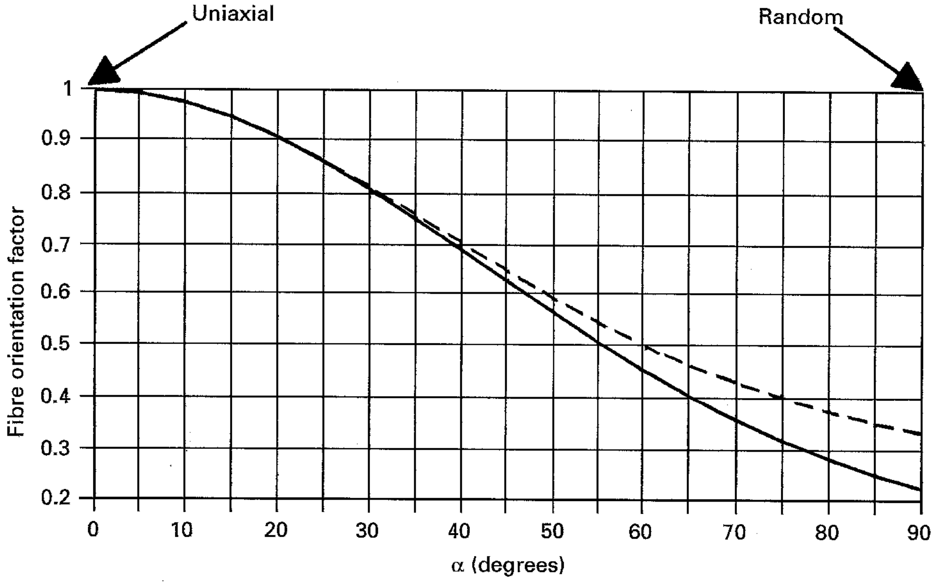

Figure 3 Fibre orientation factor as a function of α (in degrees). - - - C_{PP}; —— C_{CM}.

$$= \left[\frac{E_m}{E_f(1+\nu_m)r_f^2 \ln(R/r_f)}\right]^{1/2} \quad (32)$$

where G_m = shear modulus of the matrix, R = distance from the centre of the fibre under consideration to the ring of the nearest neighbours, E_m = Young's modulus of the matrix, and ν_m = Poisson's ratio of the matrix. Piggott [9] has shown that for a hexagonal fibre packing arrangement

$$\ln\left(\frac{R}{r_f}\right) = \frac{1}{2}\ln\left(\frac{2\pi}{3^{1/2}V_f}\right) \quad (33)$$

Using Equation 31, in Equations 18 and 27, the fibre length factors for this limiting case can be rewritten as

$$D_{PP} = \frac{1}{\bar{L}}\int_0^\infty \left\{1 - \frac{\tanh(\beta L/2)}{(\beta L/2)}\right\} L h(L)\,\mathrm{d}L$$

$$D_{CM} = \int_0^\infty \left\{1 - \frac{\tanh(\beta L/2)}{(\beta L/2)}\right\} h(L)\,\mathrm{d}L \quad (34)$$

When all the fibres in the specimen are of equal length, the two approaches result in the same fibre length factors. However, this is not the case for any other fibre length distribution function. As an example, the fibre length factors resulting from the two approaches are calculated for the following probability density function for fibre length distribution:

$$h(L) = \frac{1}{\lambda} \text{ in } 0 \leqslant L \leqslant \lambda$$

$$h(L) = 0 \text{ in } L > \lambda \quad (35)$$

The function is shown schematically in Fig. 4. An E-glass/epoxy composite with a hexagonal fibre packing arrangement is considered. The material properties for this composite are [8]: $V_f = 0.5$, $E_f = 72$ GPa, $r_f = 5$ μm, $E_m = 2.5$ GPa, $\nu_m = 0.3$. The fibre length factors D_{PP} and D_{CM} are shown in Fig. 5 as a function of λ. As can be seen, the calculation procedure outlined under the composite mechanics approach consistently underpredicts the fibre length factor and thus the force sustained by the fibres across the scan line.

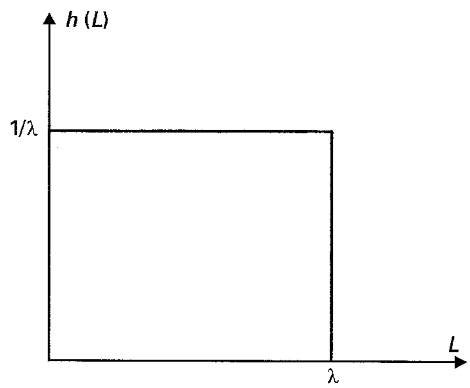

Figure 4 Schematic of the probability density function for fibre length distribution.

7. Conclusion

The theories for modulus and strength of short fibre-reinforced composite materials are based on the calculation of the force sustained by fibres crossing the scan line. The Cox theory for modulus is based on a simple but correct calculation procedure while the Fukuda–Kawata modulus theory and the Fukuda–Chou strength theory are based on an apparently incorrect procedure for the calculation of the force sustained by the fibres crossing the scan line. The error becomes clear once the fibre orientation factor and the fibre length factor for modulus are calculated according to the Cox and Fukuda–Kawata theories. The magnitude of the error depends on the fibre orientation distribution, the fibre length distribution and the magnitude of the shear lag effect.

Acknowledgements

The authors gratefully acknowledge the financial support of the National Sciences and Engineering

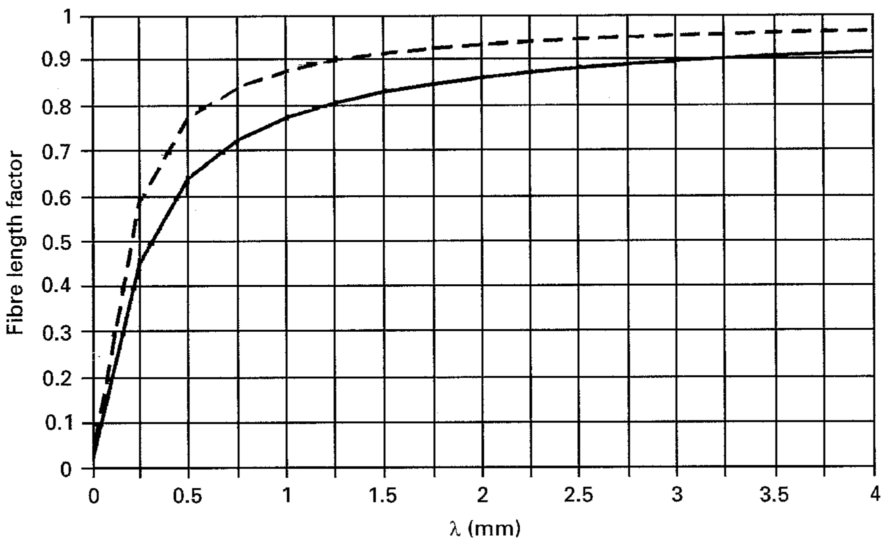

Figure 5 Fibre length factor as a function of λ (in mm) for E-glass/epoxy composite. - - - D_{PP}; —— D_{CM}.

Research Council of Canada through the Mechanical and Chemimechanical Wood Pulps Network, a National Network of Centres of Excellence. The authors also acknowledge the valuable suggestions of Dr M. R. Piggott (Department of Chemical Engineering and Applied Chemistry, University of Toronto, Toronto, Ontario, Canada M5S 1A4) regarding the content of this article.

References

1. H. L. COX, *Br. J. Appl. Phys.* **3** (1952) 72.
2. O. J. KALLMES and M. PEREZ, "*Consolidation of the Paper Web*", edited by F. Bolam (Technical Section of the British Paper and Board Makers' Association, London, UK, 1966) pp. 779–813.
3. O. J. KALLMES, G. A. BERNIER and M. PEREZ, *Paper Technol. Ind.* **18** (1977) 222.
4. *Idem, ibid.* **18** (1977) 243.
5. *Idem, ibid.* **18** (1977) 283.
6. *Idem, ibid.* **18** (1977) 328.
7. H. FUKUDA and K. KAWATA, "On Young's Modulus of Short Fibre Composites", Fibre Science and Technology, Vol. 7, pp. 207–222, 1974.
8. H. FUKUDA and T-W. CHOU, "A Probabilistic Theory of the Strength of Short-Fibre Composites With Variable Fibre Length and Orientation," Journal of Materials Science, Vol. 17, pp. 1003–1011, 1982.
9. M. R. PIGGOTT, *Load-Bearing Fibre Composites*, Pergamon Press, New York, NY, 1980, Chapter 5.

Received 12 April
and accepted 17 October 1995